Hierarchical representation of 2D polygons based on approximate skeletons

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Rapporto Tecnico No.8/00

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1. INTRODUCTION

Dealing with shape means first of all determining what "shape" is. When we think about appearance, of course we have in mind the idea of "shape" as synthesis of colour, volume, proportion, and many other characteristics of an object. But when we have to formalise the concept of shape, everything becomes very difficult. Attempts have been done since the ancient Greeks philosophers era, but a universal definition of shape hasn't been achieved so far.

Now let's restrict our interest to the computer graphics context. Through the geometric modelling process, a real object is described first as an abstract mathematical model, which will be implemented inside the computer in terms of data structure (digital model). Since we directly deal with digital models, we need at least to define descriptors suitable to capture our intuitive notion of shape.

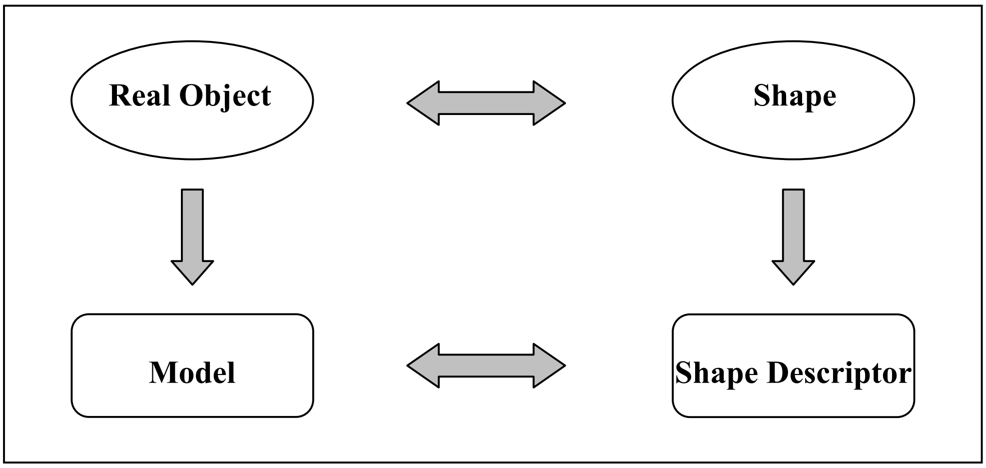


Figure 1: relations between the real object and its digital model, and between real shape and shape descriptor.

It is evident shape is an intrinsic property, which doesn't change according to the object's position in space. Thereafter we need co-ordinate-independent shape descriptors (invariant to affine transformations). Moreover, being shape an intrinsic characteristic, a good shape descriptor is also likely to be independent on data sampling density.

Another point is the following: it is so easy for human perception to capture similarity among objects, while it's not trivial at all to discriminate between similar and different models or to say "how much" two objects looks alike automatically (e.g. in object retrieval/recognition). To solve the model comparison in an intuitive way we must guarantee that objects having similar shape have similar representations either. This corresponds to the requirement for a shape descriptor to be robust, so that a small change in the object appearance leads to a small change in the description as well. In the same way, a small perturbation affecting data will change the representation by a small percentage.

The challenge of finding an optimal shape descriptor suitable for all the applications is still open; we present here a hierarchical description for 2D polygons as a powerful tool for understanding shape structure and show that this description is invariant to affine transformations, robust and sampling independent. We also briefly sketch the vertex correspondence problem in morphing as one of the possible fields of application of this method.

2. STATE OF THE ART

There are lots of works dealing with the problem of finding suitable shape description for purposes such as shape similarity and matching, recognition and classification, morphing, reconstruction, rendering and many others. Most of them use some high-level structure to describe the overall shape like modified Voronoi diagrams or distance fields for this purpose [1, 2, 3] or some kind of skeleton [4, 5]. Shapira and Rappoport in [6] use a planar graph obtained by joining the star points of each star-shaped polygon in which the original shape can be divided. This *star skeleton* represents all points in the shape's interior as well as on its boundary and defines an explicit dependence between interpolated polygon vertices relatively to a common structure. In any case the method is dedicated to blending: their approach first interpolates between two skeletons and then unfolds the star pieces from the structure to generate the intermediate shape; the problem is the interpolation algorithm works only on isomorphic structures and therefore the skeleton extraction is forced to fulfil this requirement. This means the correspondence between the shape and its representation is dependent on a second polygon which the source must be blend with.

Others are feature-based methods, such as [7] where shape is coded as a sequence of curvature primitives (bumps, corners…) detected parsing the contour. The authors also show it's possible to construct a multiresolution representation of a contour by interpreting the significant changes in curvature at various scales. This approach is robust and through multiresolution can discriminate between meaningful features and small details, but lacks in taking into account the global aspect of the shape, since no information on the interior structure is coded.

Among the existing techniques for describing polygonal shapes, the Medial Axis (MA) is generally considered the more elegant and effective one. An intuitive definition of the skeleton in the continuum was given by Blum [8], who described the skeleton extraction by analogy with a fire front which starts at the boundary of the shape and propagates isotropically towards the interior. The medial axis is defined by the locations at which the fire fronts collide.

More formally, the medial axis of a shape S is the locus of centres of all maximal discs of S, that is, those discs contained in S which are not contained in any other disc in S. Equivalently, if $B(S)$ is the boundary of S, then the medial axis of $B(S)$ is the set of points in S having at least two nearest neighbours on $B(S)$. The medial axis, together with the radius function, i.e. the distance from each point on the axis to the nearest point on the boundary, define the *medial axis* *transformation* (MAT). The power of this representation is that the shape's boundary and its MAT are equivalent and one can be computed from the other, therefore a two-dimensional object is effectively transformed into a one-dimensional graph-like structure.

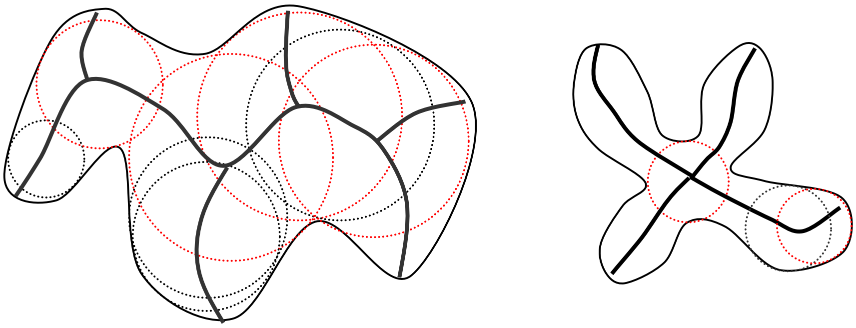


Figure 2 The Medial Axis of two simple shapes

The radius function and the graph-like structure of the MAT induce a decomposition of the shape into subparts which correspond to an intuitive notion of shape protrusion and branching. The points on the axis, may indeed be classified according to the number of corresponding nearest neighbours on the shape boundary (see figure 2): *end* points have a single contact point with the boundary, *normal* points have two contact points and *branch* points have three or more contact points. A branch point indicates a split of the shape into diverging "protrusions", while end points indicate a local maximum of curvature of the shape boundary.

Therefore, the medial axis can be seen as a graph whose nodes are the end and branch points, and whose arcs are defined by normal points. If the shape is a polygon, the MAT is a tree-like planar graph whose arcs are composed by straight-line segments and portions of parabolic curves, and algorithms have been defined to compute the MAT from the Voronoi diagram of the polygon elements in O($n$log$n$) time, where n is the number of edges in the polygon [9]. Finally, if the shape is simply-connected, then its medial axis graph is a tree, while cycles will appear in the MAT around each hole in the shape.

The Medial Axis seems to be a good shape descriptor since it is a global representation of a polygon and also it is co-ordinate independent; unfortunately it is sensitive on tiny perturbations, so that a small change in the polygon boundary will produce an extra edge in the graph (see fig.3).

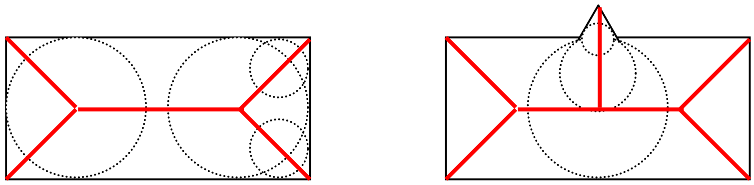


Figure 3 :extra edge in the MAT caused by a tiny change in the boundary

3. THE APPROXIMATE SKELETON

To describe the shape of a two-dimensional polygon the computation of an approximate skeleton has been experimented to be an efficient solution which still preserves the strength of the original one and does not present more disadvantages. In other words, for the sake of shape description, the approximate algorithm works as well as the MAT.

The idea of approximate skeleton has been used in the field of surface reconstruction, as a tool for deriving hints on the shape behaviour from a set of surface contours [10, 11]. Intuitively, the approximate medial axis is generated using the constrained Delaunay triangulation of the polygon as support structure and then by tracing medial segments of the inner triangles, according to rules which take into account the number of constrained edges of each triangle. In this way, a kind of skeleton is produced which globally has the same descriptive effect of the real medial axis.

More precisely, let S be a simply-connected polygonal shape having as boundary the polygon $P=\{P_i \ / \ i=1\ldots n, P_i \in R^2\}$; let also $T_D(P)$ be the Delaunay triangulation of S constrained to its boundary [12]. The triangles of $T_D(P)$ may have either one or two constrained edges, i.e. edges lying on the boundary P, or may have three non-constrained internal edges (see figure 4).

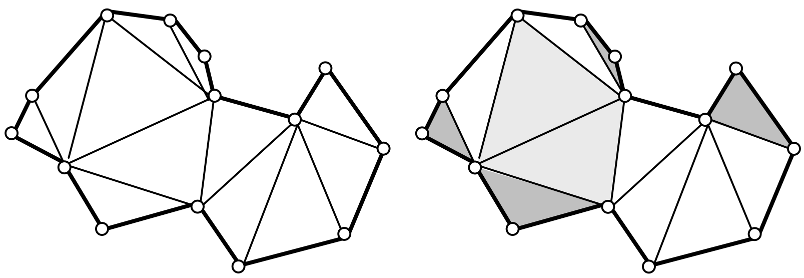


Figure 4: morphological decomposition of a polygon: branching triangles and terminal triangles

Triangles with three inner edges define branch sites of the skeleton, triangles with two edges on the boundary identify terminal nodes of the skeleton, and finally triangles having only one edge on the boundary contribute to an arc of the skeleton. The AS can be easily traced using medial lines, starting either from branch sites or terminal nodes as follows. With reference to figure 5, where the polygon P and its constrained Delaunay triangulation are depicted, let us start the process with any triangle T_i identifying a terminal node of AS. If v is the vertex of T_i shared by the two boundary edges e_1 and e_2, then the first part of the AS arc ending in v is defined by (v,p) where p is the middle point of the opposite edge e_3. Now, the delineation of the AS continues in the adjacent triangle T_j according to its type:

- T_j *has no boundary edges:* a branching in the AS occurs, which causes a branching node and two new arcs being defined; the branching node is represented by the centre of mass of T_j , c, which is joined to the middle points of its three edges (points q and r in figure 5); the tracing continues recursively on the two newly generated arcs, (c,p) and (c,r).
- T_j *has 1 boundary edge*: the current arc goes on, by joining the previous one with the middle point of the next non-boundary edge as, for example, (r,t) in figure 4.
- T_j *has 2 boundary edges:* this condition implies that the current AS arc is terminated, by drawing the segment joining the current arc portion to the vertex common to the two boundary edges.

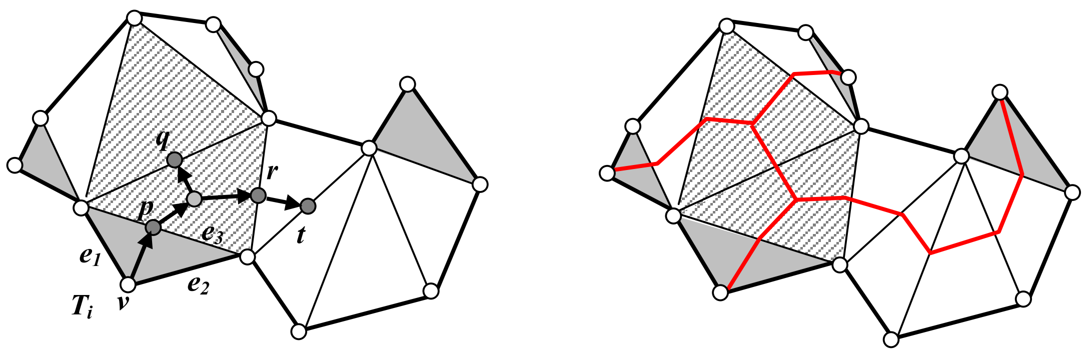


Figure 5: the AS construction

Therefore, the AS can be formally defined as follows.

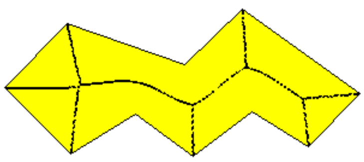
Definition 1:

Given a simply-connected polygonal shape S, with boundary defined by the polygon $P=\{P_i$ / $i=1\ldots n$, $P_i \in R^2\}$, and given the Delaunay triangulation of S constrained to its boundary $T_D(P)$, the approximate skeleton of S is the graph $AS(S) = (N,A)$ defined by the node set $N= \{T_i / T_i \in T_D(P)$ and T_i has either no constrained edges or two constrained edges$\}$ and by the arc set $A=\{(T_i,T_j)$ / there is a medial path in $T_D(P)$ between T_i and $T_j\}$

The algorithm for extracting AS has to be slightly modified if we have to compute the skeleton of a non-simply connected polygon. In this case, a starting triangle with two boundary edges may not even exist. If a triangle exists with no edges on the boundary, then the process can start by generating a branching node with degree three in the skeleton, and by proceeding recursively on the three generated directions; otherwise, the construction starts from any triangle with one edge on the boundary, with the creation of a dummy node with degree two, which will be deleted once the skeleton delineation is completed.

2.3 Differences and properties of the MAT and the AS

With respect to the MAT, the AS behaves slightly differently: first of all, the AS is a planar graph having at most degree three, and this property will be used in the similarity matching step; the AS is combinatorially simpler than the MAT, in the sense that it has fewer arcs and nodes than the MAT. In figure 5, an example is shown of MAT and AS computed on the same polygon. The decomposition induced by the AS is *equivalent* to that induced by the MAT.



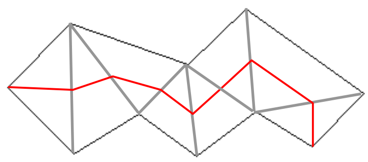


Figure 6: The MAT and the AS for the same polygon

With reference to figure 7, each arc ending in a terminal node identifies a *protrusion*-like feature of the shape (see the portions of shape coloured in grey); each arc between two branching nodes defines an *inner* part of the shape. Inner components of the shape may be further distinguished as a *normal* inner portion, as the pale orange triangles in 7(a), or in a *special* inner portions around holes in the shape, as the pale green triangles in 7(b).

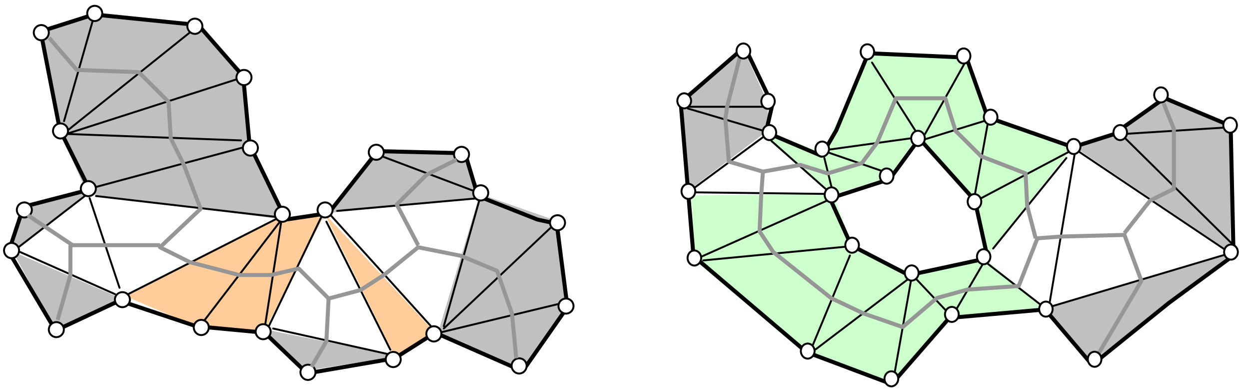


Figure 7 :The decomposition induced by the AS: protrusions are coloured in grey and normal inner components in light orange (a); special inner components around a hole are coloured in light green (b).

The AS can be therefore easily “embedded” in a geometric context, by attaching to each node and arc the shape components they represent. For example, in [13] we consider as attached to each terminal node the set of triangles traversed by the medial path from the branching node to the terminal one, and use measures like the area of the component, the medial path length and the curvature at the terminal node to match two skeleton for shape blending purposes. Such characterisation of the shape form a rich basis for organising the AS of a polygon according to the scale of its shape components, as we are going to show in the next section.

4. CONSTRUCTION OF THE HIERARCHY

As we previously stated the AS (as well as the MAT) is sensitive to small perturbations and changes on the boundary, which produce unwanted edges in the skeleton and also make it difficult to discriminate between structural and detail features. Another work strictly related to this problem is [14], which deals with the construction of a hierarchical approximated MAT obtained from the Voronoi diagram of the boundary points (*Discrete Voronoi Medial Axis*, or DVMA). Several criteria are proposed to assign each edge of the DVMA a value reflecting its relevancy and related to the shape area spanned by the edge. Then a hierarchy is constructed by pruning the full-detail DVMA with an increasing threshold. This approach is similar to the one described in this paper, but while in [14] the threshold must be user-defined, here a fully automatic pruning procedure is implemented. Moreover, our approach guarantees that the terminal nodes of the skeleton all lay on the boundary at each level of detail; this is useful for instance for morphing purposes, where some a-priori correspondences are requested to perform a pleasant blending of a shape into another.

In [15] again an approximate MAT called *Chordal Axis* (CAT) is used to extract the morphological features of a shape. The CAT is constructed starting from a Delaunay triangulation of the polygon, in a very similar way to that just described here and in other previous works [16]. Anyway this structure is used in a successive paper [17] as a basis for a syntactic recognition and matching: a universal language for representing the features of a shape is defined, based on a finite alphabet of generic shape feature primitives. Shape exteriors are then syntactically represented as strings in this language, and associated with attribute vectors that capture metric properties of the corresponding feature; then a hierarchical shape recognition scheme is sketched.

In this section we will show a different approach, that generates an AS multiscale representation working directly on the graph structure.

Let AS_0 be the first Approximated Skeleton computed on a Polygon P. Since each edge in the AS represent a polygon component, we have to assign them a value reflecting the importance of the corresponding feature.

Definition 2: we define the importance of the arc *a* the real value

$$\rho_a = length_a * area_a$$

that is to say the product of the length and area of the crossed triangles of the edge. This measure avoid to give too much weight to long and thin features and to wide but short bumps as well. (figure 8)

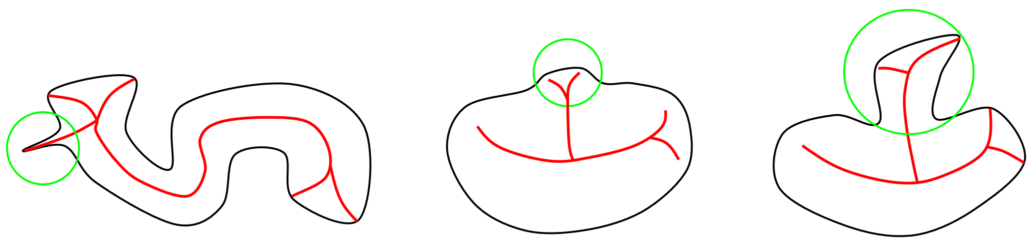


Figure 8: A thin spike (left) and a small bump (middle) on the boundary are not considered important features; a wide and long protrusion (right) is a salient feature of the shape.

Once given to each arc its importance value, the computation of the next AS at a lower level of detail works as follows:

1. A threshold ρ_T is chosen to individuate the non significant arcs at the current LOD
2. all arcs a such that $\rho_a <= \rho_T$ are deleted and the node set is updated.

4.1 Choosing the right threshold

The most delicate step of the process is undoubtedly to automatically choose the non relevant arcs which are to be removed in the next LOD. The trivial value $\rho_T = min_A\{\rho_a\}$ leads to delete at each iteration only the arc whit minimum importance. This is not useful for two reasons:

- for very complex polygons the simplification process is slow
- the problem of distinguish among features at different level of detail is still unsolved, because every arc has its own scale and is distinct from each other, no matter if its value of importance is very near to another.

  The same holds for $\rho_T = max_A\ \{\rho_a\}$ which reaches at once the final configuration whit the least number of arcs.

Therefore we want to find a threshold leading to an average "speed of simplification" between these two extrema. We tested several different criteria looking for good experimental results, and we found that what we expected from a good and "intuitive" pruning algorithm, was to guarantee that *if two arcs have similar importance, then they must belong to the same LOD.* Here is an example of what we mean and why was not trivial to achieve this purpose. Let's take as threshold the average value of importance: $\rho_T = (\Sigma_A \rho_a) / N$ and consider four terminal arcs with values $a_0 = 10$, $a_1 =20$, $a_2 = 22$, $a_3 = 28$. Intuitively, we expect the algorithm to delete a_0 in the first iteration, and then a_1 and a_2 in the second iteration, because their values are quite close with respect to the others. But the average threshold deletes a_1 together with a_0 in the first iteration, which is not reasonable.

The same problem appears with other choices of threshold like:

$$\rho_T = min_A\{\rho_a\} + \sigma$$

$$\rho_T = min_A\{\rho_a\} * \sigma$$

$$\rho_T = (max_A\ \{\rho_a\} - min_A\ \{\rho_a\}) / \sigma$$

where σ is a user-defined variable.

Therefore, our approach was to view the arc set as a value distribution. First of all, the arcs are sorted in an increasing order with respect to their importance value, and each is identified with its position in the ordered list. Let TA be the ordered set of terminal arc indices, TA = $\{0, 1, \dots, n\} \subset N$. Then consider a function

$\varphi : TA \rightarrow \Re$ such that $\varphi(i) = \rho_i$. We have defined a weakly increasing sequence. Now to find the value suitable as a threshold for pruning arcs with similar importance is related to the study of the function graph.

In the continuum case, given a function $y = f(x)$ we have that the first derivative gives a measure of the variation of y with respect to a small variation of x; the quantity $\Delta y/\Delta x$ approximates the local curve gradient. In the same way, the second derivative measures the variation of f'(x), so that the local maxima of f''(x) correspond to points where the values of f(x) change most rapidly and most differ from the previous values.

We try now to apply these notions in the discrete case, to the previously defined function φ.

Since f'(x) can be approximated by $(f(x+h) – f(x)) / h$ for values h sufficiently small, we can compute

$$(1)\ \varphi'(i) = \varphi(i+1) - \varphi(i)$$

because the smallest distance between the elements in TA is h=1. With the same observation follows the second derive can be seen as the derive of the first derive: $\varphi''(i) = \varphi'(i+1) - \varphi'(i)$ or equivalently

$$(2)\ \varphi''(i) = \varphi(i+2) - 2\varphi(i+1)+ \varphi(i).$$

This can be computed in a very simple and efficient way as differences of the arc values. Starting from a_0 we just search for the first arc a_i corresponding to a local maximum of φ'' and choose its value ρ_i as the threshold for the pruning process ρ_T. This choice is also supported by good experimental results (see figures 10 and 11).

4.2 Arc deletion and skeleton updating

Once established the appropriate threshold, the skeleton at the previous LOD is first duplicated, then all arcs with importance less than to the threshold are removed and the structure is updated according to the following rules:

- A branching node which had an arc deleted becomes an intermediate point of the new arc formed by merging the two incident arcs left
- A branching node which has had two incident arcs removed, becomes a terminal node (see fig. 9)

In all cases, the remaining arcs inherit the triangles left without arcs crossing them after the higher LOD arc removal. This will enlarge the importance of this arc in the next iteration of the pruning algorithm.

The process goes on creating ASs at lower and lower LODs, until this is no longer possible (no other local maxima for φ’’ exist).

The ordered set AS_0 , AS_1 , … , AS_K defines the multiresolution representation (MAS) of the polygon P at the level of detail 0,1, … K.

It’s important to point out that the MAS is not merely a collection of ASs at different scales, but it’s a well formed data structure in which every feature is hierarchically linked to its corresponding one in the previous and next level of detail (if it exists).

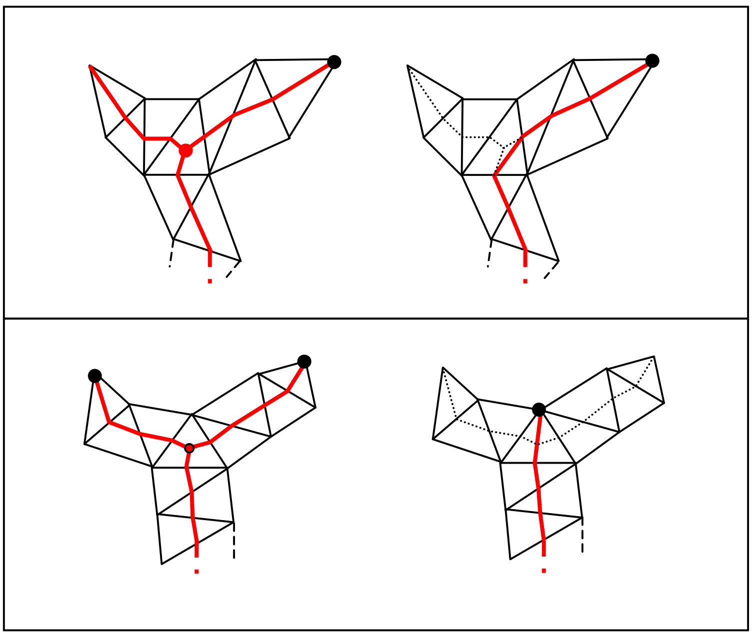


Figure 9: Effects of deletion on a branching node in the case of one or two incident arcs.

5. EXAMPLES AND RESULTS

In the previous sections we have described the AS, its properties and how to construct from the AS a multiresolution model consisting of ASs at different levels of detail. Now we are going to show how the MAS is useful in many fields of application to individuate noise effects and to recognise features at different scales.

As a first example let's examine the cross section of a shinbone(real data) made of 213 points. In figure 10 the contour with the Delaunay triangulation and the initial AS is depicted. Then the MAS is constructed in 10 iteration of the pruning algorithm: the total number of nodes (branching and terminal), the number of terminal arcs and the computed threshold for each iteration are shown in table 1. With this process, the 41 initial components of AS_0 are divided into 10 different scales. Note that the number of nodes and of terminal arcs always decreases from AS_i to AS_{i+1} , while the computed threshold always increases. Moreover the first threshold is so small to be approximated by 0; all the same 9 terminal arcs are deleted in AS_1. This means that the first iteration has removed those arcs whose length or area of crossed triangle was near to zero, and were probably inducted by noise.

Iterations	Nodes	Terminal Arcs	Computed Threshold
0	80	41	0
1	62	32	0,0055549
2	56	29	0,0163564
3	48	25	0,133894
4	40	21	0,490264
5	34	18	0,947091
6	28	15	4,24033
7	22	12	17,3947
8	14	8	81,6299
9	4	3	-

Table 1: details about the MAS construction from the shinbone data.

Another simple example just to show the robustness of this approach: imagine a shape discretized with different sampling densities; their AS are quite different even if the object they describe is the same. If we represent the two polygon by their MAS, we can find an optimal level of detail for which their skeletons are isomorphic (see figure11).

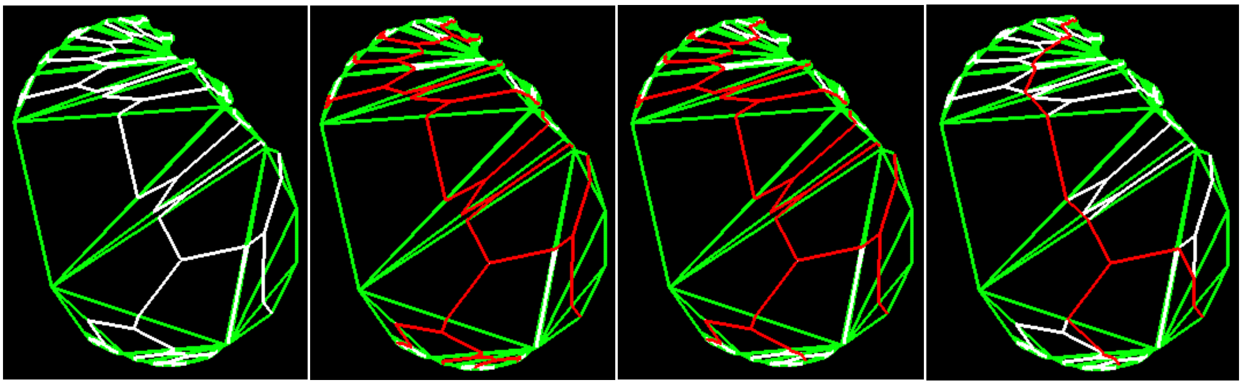


Figure 10: AS_0 , AS_6 , AS_7 and AS_9 for the shinbone data: in white is depicted the initial AS (AS_0)

6. CONCLUSION AND FUTURE WORK

We have shown that the MAS provide a powerful structural description of a shape, robust to noise and tiny changes of the boundary, and provides also a scale-based classification of the polygon's features/components. These properties make it useful in various application in which shape similarity is involved, for example in shape classification and recognition, shape retrieval, surface reconstruction from cross sections, morphing and so on. In particular we have developed this system as an automatic shape alignment tool for polygon morphing.

The term morphing is used for all methods which gradually and continuously deform a source shape in a target one while producing the in-between models. The morphing process is usually composed of warping and interpolation. The warping operation produce an approximated alignment so that the two shapes will be relatively similar; then the two warped objects are blended into one [18]. It is important to point out that in most of the well known morphing techniques the user control over the warping is still crucial in obtaining satisfactory results, for example through specifying a predetermined mapping for a set of points (point-to-point correspondence) [18, 19, 20].

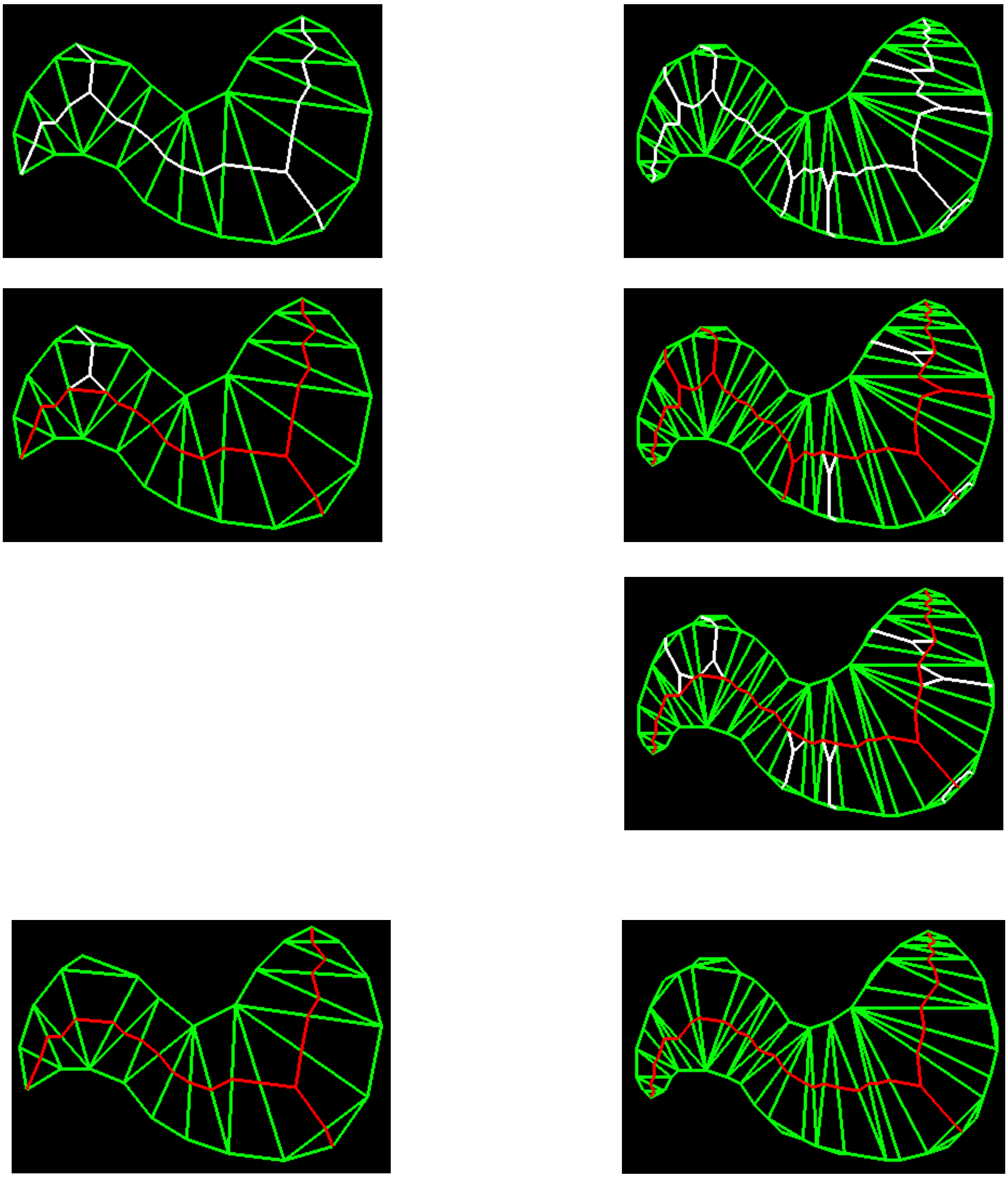


Figure 11 On the first row the same shape is represented with different number of sample points and the corresponding AS_0; in the first column one pruning step is necessary to obtain the coarsest AS for the simpler polygon; in the second column two pruning steps are performed on the more complex polygon to obtain the final skeleton; the coarsest AS for the two shapes are isomorphic (last row).

In [13] we solved the correspondence problem through an ASs matching process; unfortunately this method lead to poor results if the ASs to be matched were too different, and always needed a pruning step to overcome small perturbation or different sampling density problems. The developing of a new matching algorithm between MASs will enable to first match two simple structures at the lower level of detail, and then induce the right correspondences between features at higher scales.

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